Note on the so-called Quotient G/H in the Theory of Groups.

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The notion (see Hölder, "Zur Reduction der algebraischen Gleichungen," Math. Ann., t. XXXIV (1887), §4, p. 31) is a very important one, and it is extensively made use of in Mr. Young's paper, "On the Determination of Groups whose Order is the Power of a Prime," Amer. Math. Jour., t. XV (1893), pp. 124–178; but it seems to me that the meaning is explained with hardly sufficient clearness, and that a more suitable algorithm might be adopted, viz. instead of $G_1 = G/\Gamma_1$ I would rather write $G = \Gamma_1 . QG_1$ or $QG_1 . \Gamma_1$.

We are concerned with a group G containing as part of itself a group Γ_1 such that each element of Γ_1 is commutative with each element of G. This being so, we may write

$$G = QG_1 . \Gamma_1,$$

where QG_1 is not a group but a mere array of elements, viz. if $\Gamma_1 = (1, A_2, \ldots . A_s)$, and $QG_1 = (1, B_2, \ldots . B_t)$, then the formula is

$$G = (1, B_2, \ldots . B_t)(1, A_2, \ldots . A_s),$$

where it is to be noticed that the elements B are not determinate; in fact, if A_θ be any element of Γ_1, we may, in place of an element B, write BA_θ, for $B(1, A_2, \ldots . A_s)$ and $BA_\theta(1, A_2, \ldots . A_s)$ are, in different orders, the same elements of G.

But, G being a group, the product of any two elements of G is an element of G; viz. we thus have in general

$$B_i A_{i'} . B_j A_{j'} = B_k A_{k'}\,; \text{ that is, } B_i B_j = B_k A_{k'} A_{j'}^{-1} A_i^{-1} \quad (i, j, \text{ unequal or equal}),$$

where the B_k is a determinate element of the series $1, B_2, \ldots . B_t$, depending only on the elements B_i and B_j into the product of which it enters; and it is in

nowise affected by the before-mentioned indeterminateness of the elements B: say B_i, B_j being any two elements of the series $1, B_2, \ldots . B_t$, we have the last preceding equation wherein B_k is a determinate element of the same series.

We may imagine a set of elements $1, B_2, \ldots . B_t$ for which, B_i, B_j being any two of them and B_k a third element determined as above, we have always $B_iB_j = B_k$, that is these elements $1, B_2, \ldots . B_t$ now form a group, say the group G_1; the original elements $1, B_2, \ldots . B_t$, (which are subject to a different law of combination $B_iB_j = B_kA_{k'}A_{j'}^{-1}A_j^{-1}$, and do not form a group) are regarded as a mere array connected with this group, and so represented as above by QG_1; and the relation of the original group G to the group Γ_1 (consisting of elements commutative with those of G) and to the new group G_1 is expressed as above by the equation $G = \Gamma_1 . QG_1, = QG_1 . \Gamma_1$.

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